Example 1.3.5 *continued*
The person now draws two chips. What is the probability that the defective chip is among them?
We need to set up a new sample space containing all possibilities for drawing two chips:

$$\begin{aligned}\Omega &= \{\{g_1, g_2\}, \{g_1, g_3\}, \{g_1, d\}, \\ &\quad \{g_2, g_3\}, \{g_2, d\}, \\ &\quad \{g_3, d\}\}\end{aligned}$$

$$\begin{aligned}E &= \text{“ defective chip is among the two chips drawn”} = \\ &= \{\{g_1, d\}, \{g_2, d\}, \{g_3, d\}\}.\end{aligned}$$

Then

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{6} = 0.5.$$

Finding $P(E)$ involves counting the number of outcomes in E. Counting by hand is sometimes not feasible if Ω is large.
Therefore, we need some standard counting methods.

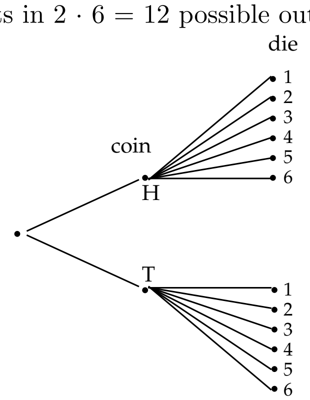
1.4 Counting Methods

1.4.1 “Multiplication Principle”

If a complex action can be broken down in a series of k components and these components can be performed in respectively $n_1, n_2, \ldots, n_k$ ways, then the complex action can be performed in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ different ways.

Sounds simple, but has enormous impact! Not to be underestimated!

Example 1.4.1
Tossing a coin, then tossing a die: results in $2 \cdot 6 = 12$ possible outcomes of the experiment.



1.4.2 Ordered Samples with Replacement

just to make sure we know what we are talking about, here are the definitions that will explain this section's title:

Definition 1.4.1 (ordered sample)
If r objects are selected from a set of n objects, and if the order of selection is noted, the selected set of r objects is called an *ordered sample.*

Definition 1.4.2 (Sampling w/wo replacement)
?? *Sampling with replacement* occurs when an object is selected and then replaced before the next object is selected.
Sampling without replacement occurs when an object is not replaced after it has been selected.

Situation:

Imagine a box with n balls in it numbered from 1 to n. We are interested in the number of ways to sequentially select k balls from the box when the same ball can be drawn repeatedly (with replacement).

This is our first application of the multiplication principle: Instead of looking at the complex action, we break it down into the k single draws. For each draw, we have n different possibilities to draw a ball.
The complex action can therefore be done in $\underbrace{n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k$ different ways.

The sample space Ω can be written as:

$$\begin{aligned} \Omega &= \{(x_1, x_2, \ldots, x_k) | x_i \in \{1, \ldots, n\}\} \\ &= \{x_1 x_2 \ldots x_k | x_i \in \{1, \ldots, n\}\} \end{aligned}$$

We already know that $|\Omega| = n^k$.

Example 1.4.2

(a) How many valid five digit octal numbers (with leading zeros) do exist?

In a valid octal number each digit needs to be between 0 and 7. We therefore have 8 choices for each digit, yielding 8^5 *different five digit octal numbers.*

(b) What is the probability that a randomly chosen five digit number is a valid octal number?

One possible sample space for this experiment would be

$$\Omega = \{x_1 x_2 \ldots x_5 | x_i \in \{0, \ldots, 9\}\},$$

yielding $|\Omega| = 10^5$.

Since all numbers in Ω *are equally likely, we can apply Thm 1.3.4 and get for the sought probability:*

$$P(\text{ “randomly chosen five digit number is a valid octal number” }) = \frac{8^5}{10^5} \approx 0.328.$$

Example 1.4.3 *Pick 3*
Pick 3 is a game played daily at the State Lottery. The rules are as follows:
Choose three digits between 0 and 9 and order them.
To win, the numbers must be drawn in the exact order you've chosen.
Clearly, the number of different ways to choose numbers in this way is $10 \cdot 10 \cdot 10 = 1000$.
odds (= probability) to win: 1/1000.

1.4.3 Ordered Samples without Replacement

Situation:

Same box as before. We are interested in the number of ways to sequentially draw k balls from the box when each ball can be drawn only once (without replacement).

Again, we break up the complex action into k single draws and apply the multiplication principle:

Draw	1st	2nd	3rd	...	kth
# of Choices	n	$(n-1)$	$(n-2)$	...	$(n-k+1)$

total choices:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

The fraction $\frac{n!}{(n-k)!}$ is important enough to get a name of its own:

Definition 1.4.3 (Permutation number)
$P(n,k) := n!/(n-k)!$ is the number of permutations of n distinct objects taken k at a time.

Example 1.4.4

(a) I only remember that a friend's (4 digit) telephone number consists of the numbers 3,4, 8 and 9.

How many different numbers does that describe?

That's the situation, where we take 4 objects out of a set of 4 objects and order them - that is $P(4,4)$!.

$$P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{24}{1} = 24.$$

(b) In a survey, you are asked to choose from seven items on a pizza your favorite three and rank them.

How many different results will the survey have at most? - $P(7,3)$.

$$P(7,3) = \frac{7!}{(7-3)!} = 7 \cdot 6 \cdot 5 = 210.$$

Variation: How many different sets of "top 3" items are there? (i.e. now we do not regard the order of the favorite three items.)

Think: The value $P(7,3)$ is the result of a two-step action. First, we choose 3 items out of 7. Secondly, we order them.

Therefore (multiplication principle!):

$$\underbrace{P(7,3)}_{\substack{\text{\# of ways to choose} \\ \text{3 from 7 and order them}}} = \underbrace{X}_{\substack{\text{\# of ways to choose} \\ \text{3 out of 7 items}}} \cdot \underbrace{P(3,3)}_{\substack{\text{\# of ways to choose} \\ \text{3 out of 3 and order them}}}$$

So:

$$X = \frac{P(7,3)}{P(3,3)} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

This example leads us directly to the next section:

1.4.4 Unordered Samples without Replacement

Same box as before.
We are interested in the number of ways to choose k balls (at once) out of a box with n balls.

As we've seen in the last example, this can be done in

$$\frac{P(n,k)}{P(k,k)} = \frac{n!}{(n-k)!k!}$$

different ways.
Again, this number is interesting enough to get a name:

Definition 1.4.4 (Binomial Coefficient)
For two integer numbers n, k with $k \leq n$ the Binomial coefficient is defined as

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}$$

Read: "out of n choose k" or "k out of n".

Example 1.4.5 *Powerball (without the Powerball)*
Pick five (different) numbers out of 49 - the lottery will also draw five numbers.
You've won, if at least three of the numbers are right.

(a) What is the probability to have five matching numbers?

Ω, the sample space, is the set of all possible five-number-sets:

$$\Omega = \{\{x_1, x_2, x_3, x_4, x_5\} | x_i \in \{1, \ldots, 49\}\}$$

$$|\Omega| = \binom{49}{5} = \frac{49!}{5!44!} = 1906884.$$

The odds to win a matching five are 1: 1 906 884 - they are about the same as to die from being struck by lightning.

(b) What is the probability to have exactly three matching numbers?

Answering this question is a bit tricky. But: since the order of the five numbers you've chosen doesn't matter, we can assume that we picked the three right numbers at first and then picked two wrong numbers.

Do you see it? That's again a complex action that we can split up into two simpler actions.

We need to figure out first, how many ways there are to choose 3 numbers out of the right 5 numbers. Obviously, this can be done in $\binom{5}{3} = 10$ ways.

Secondly, the number of ways to choose the remaining 2 numbers out of the wrong 49-5 = 44 numbers is $\binom{44}{2} = 231$.

In total, we have 10 · 231 = 2310 possible ways to choose three right numbers, which gives a probability of 11/90804 ≈ 0.0001.

Note: the probability to have exactly three right numbers was given as

$$P(\text{ "3 matching numbers" }) = \frac{\binom{5}{3}\binom{49-5}{5-3}}{\binom{49}{5}}$$

We will come across these probabilities quite a few times from now on.

(b) What is the probability to win? (i.e to have at least three matching numbers)

In order to have a win, we need to have exactly 3, 4 or 5 matching numbers. We already know the probabilities for exactly 3 or 5 matching numbers. What remains, is the probability for exactly 4 matching numbers.

If we use the above formula and substitute the 3 by a 4, we get

$$P(\text{ “4 matching numbers” }) = \frac{\binom{5}{4}\binom{49-5}{5-4}}{\binom{49}{5}} = \frac{5 \cdot 49}{\binom{49}{5}} \approx 0.000128$$

In total the probability to win is:

$$\begin{aligned} P(\text{ “win” }) &= P(\text{ “3 matching numbers” }) + P(\text{ “4 matches” }) + P(\text{ “5 matches” }) = \\ &= \frac{1 + 5 \cdot 49 + 231}{1906884} = 477 : 1906884 \approx 0.00025. \end{aligned}$$

Please note: In the previous examples we've used parentheses ( ), see definition , to indicate that the order of the elements inside matters. These constructs are called *tuples*.
If the order of the elements does not matter, we use { } - the usual symbol for *sets*.