A Paternity Paradox
Wright State University
S. L. Zabell

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Suppose a locus has three alleles, P, Q, R with frequencies p, q, r respectively ($p + q + r = 1$). If a child with an unknown mother has genotype PQ, find the possible genotypes and their probabilities for the child's father. (Asssume mating is at random and mutation does not occur.)

Solution. One has to compute the conditional probabilities

$$\frac{P(\text{father has genotype } G_1G_2, \text{child has genotype } PQ)}{P(\text{child has genotype } PQ)}.$$

If mutation does not occur, the father must have either the P or Q allele, and the possible genotypes and their probabilities are:

$$
\begin{array}{ll}
PP & \dfrac{p^2q}{2pq} = \dfrac{p}{2} \\
PQ & \dfrac{q^2p}{2pq} = \dfrac{q}{2} \\
PQ & \dfrac{2pq\left(\frac{1}{2}q + \frac{1}{2}p\right)}{2pq} = \dfrac{p+q}{2} \\
PR & \dfrac{2pr\left(\frac{1}{2}q\right)}{2pq} = \dfrac{r}{2} \\
QR & \dfrac{2qr\left(\frac{1}{2}p\right)}{2pq} = \dfrac{r}{2}.
\end{array}
$$

The *paternity index* is the ratio of these conditional probabilities to the general population frequencies for the different possible genotypes of the father:

$$\begin{aligned}
PP: &\qquad PI = \frac{p}{2}\frac{1}{p^2} = \frac{1}{2p}\\
QQ: &\qquad PI = \frac{q}{2}\frac{1}{q^2} = \frac{1}{2q}\\
PQ: &\qquad PI = \frac{p+q}{2}\frac{1}{2pq} = \frac{p+q}{4pq}\\
PR: &\qquad PI = \frac{r}{2}\frac{1}{2pr} = \frac{1}{4p}\\
QR: &\qquad PI = \frac{r}{2}\frac{1}{2qr} = \frac{1}{4q}.
\end{aligned}$$

For example, if the allele frequencies are $p = 0.52, q = 0.02, r = 0.46$, then the conditional probabilities are respectively

$$0.26, 0.01, 0.27, 0.23, 0.23$$

and the paternity indices are

$$0.96, 25, 12.98, 0.48, 12.5.$$

The (apparent) paradox: note that some of these are less than 1!!.